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HÖLDER CONTINUITY OF SOLUTIONS TO THE G-LAPLACE EQUATION INVOLVING MEASURES

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Abstract. We establish regularity of solutions to the G-Laplace equation $-\text{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu$, where μ is a nonnegative Radon measure satisfying $\mu(B_r(x_0)) \leq Cr^m$ for any ball $B_r(x_0) \subset\subset \Omega$ with $r \leq 1$ and $m > n-1-\delta \geq 0$. The function $g(t)$ is supposed to be nonnegative and C^1-continuous in $[0, +\infty)$, satisfying $g(0) = 0$, and for some positive constants δ and g_0, $\delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \forall t > 0$, that generalizes the structural conditions of Ladyzhenskaya-Ural'tseva for an elliptic operator.

1. Introduction.

Let Ω be an open bounded domain of $\mathbb{R}^n (n \geq 2)$, and μ a nonnegative Radon measure in Ω with $\mu(B_r(x_0)) \leq Cr^m$ for some constant $C > 0$ whenever $B_r(x_0) \subset\subset \Omega$. We consider the equation

$$-\Delta_G u = -\text{div}\left(\frac{g(|\nabla u|)}{|\nabla u|}\nabla u\right) = \mu \quad \text{in } \mathcal{D}'(\Omega), \tag{1}$$

where $G(t) = \int_0^t g(s)\mathrm{d}s$, $g(t)$ is a nonnegative C^1 function in $[0, +\infty)$, satisfying $g(0) = 0$ and the following structural condition

$$0 < \delta \leq \frac{tg'(t)}{g(t)} \leq g_0, \quad \forall\ t > 0,\ \delta, g_0 \text{ are positive constants.} \tag{2}$$

The structural conditions on g was introduced by Lieberman in 1991, which is a natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations (see [10]). The conditions of g imply that the operator Δ_G includes not only the $p-$Laplace operator Δ_p where $g(t) = t^{p-1}$ and $\delta = g_0 = p-1$, but also the case of a variable exponent $p = p(t) > 0$:

$$-\Delta_G u = -\text{div}\ (|\nabla u|^{p(|\nabla u|)-2}\nabla u),$$

corresponding to set $g(t) = t^{p(t)-1}$, for which (2) holds if $\delta \leq t(\ln t)p'(t) + p(t) - 1 \leq g_0$ for all $t > 0$. Another typical example of g is $g(t) = t^p\log(at+b)$ with $p, a, b > 0$ where in this case $\delta = p$ and $g_0 = p + 1$. Many other examples can be found in [2, 3, 6] etc.

Under assumption (2), G is an increasing C^2 convex function, which is an N-function satisfying the so called Δ_2-condition. Thus our class of operators will be

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considered in the setting of Orlicz spaces. We recall the definitions of Orlicz and Orlicz-Sobolev spaces together with their respective norms (see [1])

$$\begin{aligned}
&L^G(\Omega) = \{u \in L^1(\Omega);\ \int_\Omega G(|u(x)|)\mathrm{d}x < +\infty\},\\
&\|u\|_{L^G(\Omega)} = \inf\left\{k>0;\ \int_\Omega G\left(\frac{|u(x)|}{k}\right)\mathrm{d}x \le 1\right\},\\
&W^{1,G}(\Omega) = \{u \in L^G(\Omega); \nabla u \in L^G(\Omega)\},\\
&\|u\|_{W^{1,G}(\Omega)} = \|u\|_{L^G(\Omega)} + \|\nabla u\|_{L^G(\Omega)}.
\end{aligned}$$

Under the assumption (2), $W^{1,G}(\Omega)$ is a reflexive and separable Banach space (see [1]).

We shall call a solution of (1) any function $u \in W^{1,G}_{loc}(\Omega)$ that satisfies

$$\int_\Omega \frac{g(|\nabla u|)}{|\nabla u|}\nabla u\cdot\varphi\mathrm{d}x = \int_\Omega \varphi\mathrm{d}\mu \quad \forall\varphi\in\mathcal{D}(\Omega).$$

If $\mu\equiv 0$ in a domain $D\subset\Omega$, we say that u is $G-$harmonic in D.

We now introduce the regularity of the related elliptic equations involving measures. In 1994, Kilpeläinen considered the situation of the p-Laplace operator and proved that if μ satisfies $\mu(B_r)\le Cr^{n-p+\alpha(p-1)}$ for some positive constants C and $\alpha\in(0,1]$, then any solution of the p-Laplace equation

$$-\Delta_p u = -\mathrm{div}\ (|\nabla u|^{p-2}\nabla u) = \mu, \tag{3}$$

is $C^{0,\beta}_{lco}-$continuous for each $\beta\in(0,\alpha)$ (see [7]). This result was improved by Kilpeläinen and Zhong in 2002, showing that if each solution of (3) is in fact Hölder continuous with the same exponent α as the one in the assumption $\mu(B_r)\le Cr^{n-p+\alpha(p-1)}$ (see [8]). In 2010, the $p-$Laplace problem (3) was extended by Lyaghfouri to the case with variable exponents, i.e., considering

$$-\mathrm{div}\ (|\nabla u|^{p(x)-2}\nabla u) = \mu. \tag{4}$$

Under certain assumptions on the function $p(x)$ and the assumption $\mu(B_r)\le Cr^{n-p(x)+\alpha(p(x)-1)}$ for some positive constants C and $\alpha\in(0,1]$, the author proved that any boundeded solution of (4) is $C^{0,\alpha}_{loc}-$continuous with the same exponent α (see [11]).

When focusing on the problem governed by $G-$Laplacian, if $\mu(B_r(x_0))\le Cr^m$ with $m\in[n-1,n)$, Challal and Lyaghfouri proved that any solution of (1) is $C^{0,\alpha}_{loc}-$continous with $\alpha=\frac{m-n+1+\delta}{1+g_0}$(see [3]). Particularly, if $m=n-1$, then any solution is $C^{0,\alpha}_{loc}$-continuous for any $\alpha\in(0,\frac{\delta}{g_0})$ (see Theorem 3.3 in [3]). In 2011, these regularities were improved by Challal and Lyaghfouri in [5], showing that any local bounded solution of (1) is $C^{0,\alpha}_{loc}-$continous for any $\alpha\in(0,\frac{m-n+1+\delta}{g_0})$ provided that $m>n-1-\delta$. Note that under the assumption of non-decreasing monotonicity on $\frac{g(t)}{t}$, Zheng, Feng and Zhang obtained local $C^{1,\alpha}-$continuity of solutions for $m>n$ and local Hölder continuity with small exponents for some $m<n$ in 2015 (see [14]).

In this paper, we continue the work of Challal, Lyaghfouri and Zheng et al. by improving the regularity of solutions of the equation (1). Particularly, we can prove the $C^{0,\alpha}_{loc}-$continuity of solutions for any $\alpha\in(0,1)$ if $m=n-1$. More precisely, for any $m>n-1-\delta$ and without any monotonicity assumption on $\frac{g(t)}{t}$, we have the following result.

Theorem 1.1. *Assume that μ satisfies (1) with $m > n-1-\delta \geq 0$. Then we have*

(i) If $m > n$, then $u \in C^{1,\alpha}_{loc}(\Omega)$ for any $\alpha \in (0, \min\{\frac{\sigma}{1+g_0}, \frac{m-n}{2(1+g_0)}\})$, where σ is the same as in Lamma 2.4.

(ii) If $m \in [n-1, n)$, then $u \in C^{0,\alpha}_{loc}(\Omega)$ for any $\alpha \in (0,1)$.

(ii) If $n-1-\delta < m < n-1$, then $u \in C^{0,\alpha}_{loc}(\Omega)$ for any $\alpha \in (0, \frac{m-n+1+\delta}{\delta})$.

Remark 1. In [7], the author proved for the $p-$Laplacin problem that $u \in C^{0,\alpha}_{loc}(\Omega)$ for any $\alpha \in (0,1)$ provided $m = n-1$. In this paper we not only improve the results of [3, 5] and [14], but also extend the problem in [7] to general equations which governed by a large class of degenerate and singular elliptic operators.

2. **Preliminary.**

In this section, we state some auxiliary results which will be used throughout this paper. We begin with some properties of the function G.

Lemma 2.1 ([13, Lemma 2.1, Remark 2.1])**.** *Function G has the following properties:*

(G_1) *G is convex and C^2.*

(G_2) $\frac{tg(t)}{1+g_0} \leq G(t) \leq tg(t)$, *for all $t \geq 0$.*

(G_3) $\min\{s^{\delta+1}, s^{g_0+1}\}\frac{G(t)}{1+g_0} \leq G(st) \leq (1+g_0)\max\{s^{\delta+1}, s^{g_0+1}\}G(t)$.

(G_4) $G(a+b) \leq 2^{g_0}(1+g_0)(G(a)+G(b))$ *for all $a, b > 0$.*

For much more properties of G and problems governed by the operator Δ_G, please see [2, 3, 4, 5, 6, 13, 14, 15, 16] etc.

The following lemmas are some properties of $G-$harmonic functions. Throughout this paper, without special states, by B_R and B_r we denote the balls contained in Ω with the same center. Moreover, $B_r \subset\subset B_R \subset\subset \Omega$.

Lemma 2.2 ([13, Theorem 2.3])**.** *Assume $u \in W^{1,G}(\Omega)$. Let h be a weak solution of*

$$\Delta_G h = 0 \quad in\ B_R, \qquad h - u \in W^{1,G}_0(B_R),$$

then

$$\int_{B_R} (G(|\nabla u|) - G(|\nabla h|))\,dx \geq C\bigg(\int_{A_2} G(|\nabla u - \nabla h|)\,dx + \int_{A_1} \frac{g(|\nabla u|)}{|\nabla u|}|\nabla u - \nabla h|^2\,dx\bigg),$$

where $A_1 = \{x \in B_R; |\nabla u - \nabla h| \leq 2|\nabla u|\}$, $A_2 = \{x \in B_R; |\nabla u - \nabla h| > 2|\nabla u|\}$ and $C = C(\delta, g_0) > 0$.

Lemma 2.3 ([13, Lemma 2.7])**.** *Let $h \in W^{1,G}(\Omega)$ be a weak solution of $\Delta_G h = 0$. Then $h \in C^{1,\alpha}(\Omega)$. Moreover, there exists $C = C(n, \delta, g_0) > 0$ such that for every ball $B_r \subset\subset \Omega$ and every $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta, g_0, \|h\|_{L^\infty(B_{\frac{2}{3}r}(x_0))}) > 0$ such that*

$$\int_{B_r} G(|\nabla h|)\,dx \leq Cr^\lambda.$$

Let $(u)_r = \frac{1}{|B_r|}\int_{B_r} u\,dx$ be the average value of u on the ball B_r, we have

Lemma 2.4 (Comparison with G-harmonic functions [14, Lemma 3.1])**.** *Assume $u \in W^{1,G}(B_R)$. Let $h \in W^{1,G}(B_R)$ be a weak solution of $\Delta_G h = 0$ in B_R. Then there exists $\sigma \in (0,1)$ and $C = C(n, \delta, g_0) > 0$ such that for each $0 < r \leq R$, there holds*

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)\,dx \leq C\left(\frac{r}{R}\right)^{n+\sigma} \int_{B_R} G(|\nabla u - (\nabla u)_R|)\,dx + C\int_{B_R} G(|\nabla u - \nabla h|)\,dx.$$

Lemma 2.5 ([9, Lemma 2.7])**.** *Let $\phi(s)$ be a non-negative and non-decreasing function. Suppose that*

$$\phi(r) \leq C_1 \left(\frac{r}{R}\right)^{\alpha} \phi(R) + C_1 R^{\beta},$$

for all $r \leq R \leq R_0$, with α, β and C_1 positive constants. Then, for any $\tau < \min\{\alpha, \beta\}$, there exists a constant $C_2 = C_2(C_1, \alpha, \beta, \tau)$ such that for all $r \leq R \leq R_0$ we have

$$\phi(r) \leq C_2 r^{\tau}.$$

3. Proof of Theorem 1.1.

Lemma 3.1. *Assume $u \in W^{1,G}(\Omega)$. Let $B_R \subset\subset \Omega$ and $h \in W^{1,G}(B_R)$ be a weak solution of*

$$\Delta_G h = 0 \quad \text{in } B_R, \quad h - u \in W_0^{1,G}(B_R).$$

Then for any $\lambda \in (0, n)$, there exists $C = C(\lambda, n, \delta, g_0, \|u\|_{L^\infty(B_{2R/3})}) > 0$ such that

$$\int_{B_R} G(|\nabla u - \nabla h|)\,dx \leq CR^m + CR^{\frac{m+\lambda}{2}},$$

where λ is the same as in Lemma 2.3.

Proof. Firstly, convexity of G gives

$$\begin{aligned}
\int_{B_R} (G(|\nabla u|) - G(|\nabla h|))\mathrm{d}x &\leq \int_{B_R} \frac{g(|\nabla u|)}{|\nabla u|} \nabla u (\nabla u - \nabla h)\mathrm{d}x \\
&= \int_{B_R} (u - h)\mathrm{d}\mu \qquad (5)\\
&\leq C\mu(B_R) \\
&\leq CR^m, \qquad (6)
\end{aligned}$$

where we used the boundedness of u which forces h to be bounded too.

Let be A_1 and A_2 be defined as in Lemma 2.2. By Lemma 2.2, there exists a constant $C = C(\delta, g_0) > 0$ such that

$$\int_{B_R} (G(|\nabla u|) - G(|\nabla h|))\mathrm{d}x \geq C \int_{A_2} G(|\nabla u - \nabla h|)\mathrm{d}x \qquad (7)$$

and

$$\int_{B_R} (G(|\nabla u|) - G(|\nabla h|))\mathrm{d}x \geq C \int_{A_1} \frac{g(|\nabla u|)}{|\nabla u|} |\nabla u - \nabla h|^2 \mathrm{d}x. \qquad (8)$$

By (G_2), $\frac{G(t)}{t}$ is increasing in $t>0$. It follows from (G_2), (G_3), (6), (8) and Lemma 2.2 that

$$\begin{aligned}
\int_{A_1} G(|\nabla u-\nabla h|)\mathrm{d}x &= \int_{A_1} \frac{G(|\nabla u-\nabla h|)}{|\nabla u-\nabla h|}(|\nabla u-\nabla h|)\mathrm{d}x \\
&\le \int_{A_1} \frac{G(2|\nabla u|)}{2|\nabla u|}|\nabla u-\nabla h|\mathrm{d}x \\
&\le C\int_{A_1} \frac{G(|\nabla u|)}{|\nabla u|}|\nabla u-\nabla h|\mathrm{d}x \\
&= C\int_{A_1} \frac{\sqrt{G(|\nabla u|)}}{|\nabla u|}|\nabla u-\nabla h|\cdot\sqrt{G(|\nabla u|)}\mathrm{d}x \\
&\le C\Big(\int_{A_1} \frac{G(|\nabla u|)}{|\nabla u|^2}|\nabla u-\nabla h|^2\mathrm{d}x\Big)^{\frac12}\Big(\int_{A_1} G(|\nabla u|)\mathrm{d}x\Big)^{\frac12} \\
&\le C\Big(\int_{A_1} \frac{g(|\nabla u|)|\nabla u|}{|\nabla u|^2}|\nabla u-\nabla h|^2\mathrm{d}x\Big)^{\frac12}\Big(\int_{A_1} G(|\nabla u|)\mathrm{d}x\Big)^{\frac12} \\
&= C\Big(\int_{A_1} \frac{g(|\nabla u|)}{|\nabla u|}|\nabla u-\nabla h|^2\mathrm{d}x\Big)^{\frac12}\Big(\int_{B_R} G(|\nabla u|)\mathrm{d}x\Big)^{\frac12} \\
&\le C\Big(\int_{B_R} (G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\Big)^{\frac12}\Big(\int_{B_R} G(|\nabla u|)\mathrm{d}x\Big)^{\frac12} \\
&= C\Big(\int_{B_R} (G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\Big)^{\frac12} \\
&\quad\cdot\Big(\int_{B_R} (G(|\nabla u|)-G(|\nabla h|)+G(|\nabla h|))\mathrm{d}x\Big)^{\frac12} \\
&\le C\int_{B_R} (G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x \\
&\quad+C\Big(\int_{B_R} (G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x\Big)^{\frac12}\Big(\int_{B_R} G(|\nabla h|)\mathrm{d}x\Big)^{\frac12}, \\
&\le CR^m+CR^{\frac{m+\lambda}{2}},
\end{aligned} \tag{9}$$

where in the last inequality but one we used $(a+b)^\gamma \le a^\gamma+b^\gamma$ for any $a\ge 0, b\ge 0$ and $\gamma\in(0,1)$. By (7) and (9), we have

$$\begin{aligned}
\int_{B_R} G(|\nabla u-\nabla h|)\mathrm{d}x &= \int_{A_2} G(|\nabla u-\nabla h|)\mathrm{d}x+\int_{A_1} G(|\nabla u-\nabla h|)\mathrm{d}x \\
&\le C\int_{B_R} (G(|\nabla u|)-G(|\nabla h|))\mathrm{d}x+CR^m+CR^{\frac{m+\lambda}{2}} \\
&\le CR^m+CR^{\frac{m+\lambda}{2}}.
\end{aligned}$$

□

Proof of Theorem 1.1. Let h be a G-harmonic function in B_R that agrees with u on the boundary, i.e.,

$$\operatorname{div}\frac{g(|\nabla h|)}{|\nabla h|}\nabla h=0 \text{ in } B_R \quad\text{and}\quad h-u\in W_0^{1,G}(B_R).$$

By Lemma 2.4 and Lemma 3.1, for any $r \leq R$ there holds

$$\begin{aligned}
&\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \\
&\leq C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + C\int_{B_R} G(|\nabla u - \nabla h|)\mathrm{d}x \\
&\leq C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + CR^m + CR^{\frac{m+\lambda}{2}},
\end{aligned}$$

where λ is an arbitrary constant in $(0, n)$.

(i) If $m > n$, then we have

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \leq C\left(\frac{r}{R}\right)^{n+\sigma}\int_{B_R} G(|\nabla u - (\nabla u)_R|)\mathrm{d}x + CR^{\frac{m+\lambda}{2}}.$$

Since $m > n$ and λ is an arbitrary constant in $(0, n)$, one may have choose λ satisfying $\frac{m+\lambda}{2} > n$. In view of Lemma 2.5, we conclude that for any $\tau < \min\{\sigma, \frac{m+\lambda}{2} - n\}$ there holds

$$\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \leq Cr^{n+\tau}, \quad \forall\, r \leq R. \tag{10}$$

Now we claim that

$$\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x \leq Cr^{n+\frac{\tau}{1+g_0}}, \quad \forall\, r \leq R. \tag{11}$$

Indeed, for r satisfying $r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x \leq r^{\frac{\tau}{1+g_0}}$, (11) holds with $C = 1$. Now for r satisfying $r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x > r^{\frac{\tau}{1+g_0}}$, we infer from the increasing monotonicity of $\frac{G(t)}{t}$ in $t > 0$,

$$\frac{G\big(r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x\big)}{r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x} \geq \frac{G\big(r^{\frac{\tau}{1+g_0}}\big)}{r^{\frac{\tau}{1+g_0}}}.$$

It follows from (G_2) and (G_3)

$$\begin{aligned}
\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x &\leq \frac{r^{n+\frac{\tau}{1+g_0}}}{G\big(r^{\frac{\tau}{1+g_0}}\big)} G\left(r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x\right) \\
&\leq \frac{Cr^{n+\frac{\tau}{1+g_0}}}{r^{\tau}G(1)} G\left(r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x\right) \\
&\leq \frac{Cr^{n+\frac{\tau}{1+g_0}}}{r^{\tau}g(1)} G\left(r^{-n}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x\right).
\end{aligned} \tag{12}$$

Note that convexity of G and (10) implies that

$$G\left(\frac{1}{|B_r|}\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x\right) \leq \frac{1}{|B_r|}\int_{B_r} G(|\nabla u - (\nabla u)_r|)\mathrm{d}x \leq Cr^{\tau}. \tag{13}$$

By (G_3), (12) and (13), one may get

$$\int_{B_r} |\nabla u - (\nabla u)_r|\mathrm{d}x \leq Cr^{n+\frac{\tau}{1+g_0}},$$

where C depends only on $g(1), g_0$ and the volume of the unit ball. Now we have proven that (11) holds for any $r \leq R$. Thus $u \in C^{1,\frac{\tau}{1+g_0}}_{loc}(\Omega)$ by Campanato's

embedding Theorem. Due to the arbitrary of $\lambda \in (0,n)$, we can conclude (i) of Theorem 1.1 by letting $\lambda \to n$.

(ii) If $m \in [n-1,n]$, we only prove for $m = n-1$ due to the fact $\mu(B_r) \le Cr^m \le Cr^{n-1}$ with small r. By (G_4), Lemma 2.3 and Lemma 3.1, we infer

$$\begin{aligned}\int_{B_r} G(|\nabla u|)\mathrm{d}x &\le C\int_{B_r} G(|\nabla u - \nabla h|)\mathrm{d}x + C\int_{B_r} G(|\nabla h|)\mathrm{d}x\\ &\le Cr^m + Cr^{\frac{m+\lambda}{2}} + Cr^\lambda\\ &\le Cr^m,\end{aligned}$$

where in the last inequality we let $n > \lambda > n-1 = m$.

We claim that for any $r \le R < 1$ with $B_R \subset\subset \Omega$ and some positive constant C independent of r, there holds

$$\int_{B_r} |\nabla u|\mathrm{d}x \le Cr^{n-1+\alpha_0}, \tag{14}$$

with some $\alpha_0 \in (0,1)$.

Indeed, for $r \le R$ satisfying

$$r^{-n+1-\alpha_0}\int_{B_r} |\nabla u|\mathrm{d}x \le 1, \tag{15}$$

(14) holds with $C = 1$. For $r \le R$ satisfying

$$r^{-n+1-\alpha_0}\int_{B_r} |\nabla u|\mathrm{d}x \ge 1,$$

due to the increasing monotonicity of $F(t) = G(t) - G(1)t$ in $t \ge 1$, it follows

$$G\bigg(r^{-n+1-\alpha_0}\int_{B_r} |\nabla u|\mathrm{d}x\bigg) \ge G(1)\cdot r^{-n+1-\alpha_0}\int_{B_r} |\nabla u|\mathrm{d}x.$$

Then we have

$$\begin{aligned}\int_{B_r} |\nabla u|\mathrm{d}x &\le Cr^{n-1+\alpha_0}(r^{1-\alpha_0})^{1+\delta}G\bigg(r^{-n}\int_{B_r}|\nabla u|\mathrm{d}x\bigg)\\ &\le Cr^{n-1+\alpha_0}\cdot(r^{1-\alpha_0})^{1+\delta}\frac{1}{|B_r|}\int_{B_r} G(|\nabla u|)\mathrm{d}x\\ &\le Cr^{n-1+\alpha_0+(1-\alpha_0)(1+\delta)}\cdot r^{-n}\cdot r^m\\ &= Cr^{n-1+\alpha_0+(1-\alpha_0)(1+\delta)+m-n}.\end{aligned} \tag{16}$$

Combining (15) and (16), we may choose $\alpha_0 = \alpha_0 + (1-\alpha_0)(1+\delta) + m - n$, i.e., $\alpha_0 = 1 - \frac{n-m}{1+\delta}$ such that (14) holds for all $r \le R$.
For $m = n-1$, we conclude that $u \in C^{0,\alpha_0}_{loc}(\Omega)$ by Morrey Theorem (see page 30, [12]) with $\alpha_0 = \frac{\delta}{1+\delta}$.

Note that $\inf_{B_r} u \le \inf_{B_r} h$ (see the proof of Theorem 3.3 in [3]). Then by (5) and Lemma 2.3, we have for λ larger than $m+\alpha_0$

$$\begin{aligned}\int_{B_r} G(|\nabla u|)\mathrm{d}x &\le \int_{B_r}(u-h)\mathrm{d}\mu + \int_{B_r} G(|\nabla h|)\mathrm{d}x\\ &\le (\sup_{B_r} u - \inf_{B_r} h)\mu(B_r) + \int_{B_r} G(|\nabla h|)\mathrm{d}x\\ &\le (\sup_{B_r} u - \inf_{B_r} u)\mu(B_r) + \int_{B_r} G(|\nabla h|)\mathrm{d}x\\ &\le C\mathrm{osc}(u,B_r)r^m + Cr^\lambda\\ &\le Cr^{\alpha_0+m} + Cr^\lambda\\ &\le Cr^{m+\alpha_0},\end{aligned}$$

where $\mathrm{osc}(u,B_r) = \sup_{B_r} u - \inf_{B_r} u$. Arguing as (14), we get $u \in C^{0,\alpha_1}_{loc}(\Omega)$ with

$$\alpha_1 = 1 - \frac{n-(m+\alpha_0)}{1+\delta} = \frac{\delta}{1+\delta} + \frac{\alpha_0}{1+\delta}.$$

Repeating this process, we get $u \in C^{0,\alpha_k}_{loc}(\Omega)$ with

$$\alpha_k = \frac{\delta}{1+\delta} + \frac{\alpha_{k-1}}{1+\delta}.$$

Finally, we have $\alpha_k = \frac{\alpha_0}{(1+\delta)^k} + \delta\Sigma_{j=1}^k \frac{1}{(1+\delta)^j}$, which leads to $\lim_{k\to\infty}\alpha_k=1$, and the result follows.

(iii) If $n-1-\delta < m < n-1$, checking the proof and repeating the process as above, we may get $\alpha_0 = 1 - \frac{n-m}{1+\delta}$, $\alpha_1 = \frac{1+\delta+m-n}{1+\delta} + \frac{\alpha_0}{1+\delta}$, ..., $\alpha_k = \frac{1+\delta+m-n}{1+\delta} + \frac{\alpha_{k-1}}{1+\delta}$. Finally, one has $u \in C^{0,\alpha}_{loc}(\Omega)$ for any $\alpha \in (0, \frac{1+\delta+m-n}{\delta})$. $\square$

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